Question 1. Evaluate the following integrals.

a) $\displaystyle\int \frac{\sqrt[3]{x^5}+\sqrt[5]{x^3}}{\sqrt[3]{x}}\,dx$

b) $\displaystyle\int \frac{d}{dx}(x^2+e^x)\,dx$

c) $\displaystyle\int e^{-\ln x}\,dx$

Answer 1.

a) $$\int \frac{\sqrt[3]{x^5}+\sqrt[5]{x^3}}{\sqrt[3]{x}}\,dx = \int x^{4/3}\,dx + \int x^{4/15}\,dx$$

$$= \frac{3}{7}x^{7/3} + C_1 + \frac{15}{19}x^{19/15} + C_2$$

$$= \frac{3}{7}\sqrt[3]{x^7} + \frac{15}{19}\sqrt[15]{x^{19}} + C \;,\; C = C_1 + C_2$$

b) $$\int \frac{d}{dx}(x^2+e^x)\,dx = x^2 + e^x + C$$

c) $$\int e^{-\ln x}\,dx = \int e^{\ln x^{-1}}\,dx = \int x^{-1}\,dx$$

$$= \int \frac{1}{x}\,dx$$

$$= \ln|x| + C$$

Question 2. Find the following integrals.

a) $\displaystyle\int_0^{\sqrt{\ln 2}} 2xe^{x^2}\,dx$

b) $\displaystyle\int \frac{e^t}{3e^t+e^{2t}+2}\,dt$

c) $\displaystyle\int_0^1 \frac{x^2-1}{x^{-1}(x^4-2\sqrt{x^4}+3)^2}\,dx$

Answer 2.

a) $$\int_0^{\sqrt{\ln 2}} 2x\,e^{x^2}dx = \int_0^{\ln 2} e^u\,du = e^u\Big|_0^{\ln 2} = e^{\ln 2} - e^0 = 2-1=1$$

$u = x^2 \Rightarrow du = 2x\,dx$

$x=0 \Rightarrow u=0$, $x=\sqrt{\ln 2} \Rightarrow u=\ln 2$

b) $$\int \frac{et}{3et+e^{2t}+2}\,dt = \int \frac{et}{e^{2t}+3e^t+2}\,dt \quad ; \quad e^t = y \Rightarrow dy = e^t dt$$

$$= \int \frac{dy}{y^2+3y+2} = \int \frac{dy}{y+1} - \int \frac{dy}{y+2}$$

$$= \ln|y+1| - \ln|y+2| + C$$

$$= \ln\left|\frac{y+1}{y+2}\right| + C$$

$$= \ln\left|\frac{et+1}{et+2}\right| + C$$

c) $$\int_0^1 \frac{(x^2-1)\,dx}{x^{-1}(x^4-2\sqrt{x4}+3)^2} = \int_0^1 \frac{x(x^2-1)}{(x^4-2x^2+3)^2}\,dx = \int \frac{\frac{du}{4}}{u^2}$$

$x^4-2x^2+3 = u \Rightarrow du = (4x^3-4x)\,dx$
$= 4x(x^2-1)\,dx$

$$= \frac{1}{4}\int u^{-2}\,du$$

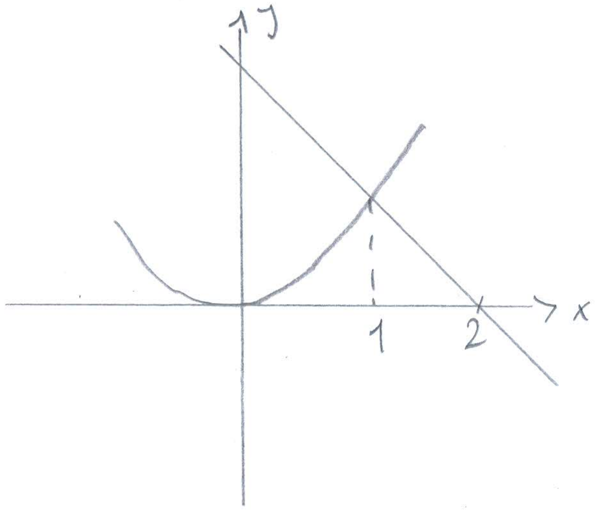
$$= \frac{1}{4}\,\frac{u^{-1}}{-1}$$

$$= -\frac{1}{4}\left(\frac{1}{x^4-2x^2+3}\right)\Big|_0^1$$

$$= -\frac{1}{4}\left(\frac{1}{1-2+3} - \frac{1}{3}\right) = -1/24$$

Question 3. Find the area bounded by the graphs of $x = \sqrt{y}$, $y = 2 - x$ and $y = 0$.

Answer 3.



$$A = \int_0^1 (x^2 - 0)\,dx + \int_1^2 (2 - x - 0)\,dx$$

$$= \int_0^1 x^2\,dx + \int_1^2 (2-x)\,dx$$

$$= \frac{1}{3}x^3\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^2$$

$$= \frac{5}{6}$$

Question 4. Determine whether the following integrals converge or diverge.

a) $\displaystyle\int_{-\infty}^{-2} \frac{2}{x^2-1}\, dx$

b) $\displaystyle\int_{1}^{2} \frac{1}{x(\ln x)^p}\, dx$, where $p \in \mathbb{R}$.

Answer 4.

a) $$\int_{-\infty}^{-2} \frac{2}{x^2-1}\, dx = \int_{-\infty}^{-2} \frac{dx}{x-1} - \int_{-\infty}^{-2} \frac{dx}{x+1}$$

$$= \lim_{b \to -\infty} \int_{b}^{-2} \frac{dx}{x-1} - \lim_{b \to -\infty} \int_{b}^{-2} \frac{dx}{x+1}$$

$$= \lim_{b \to -\infty} \ln|x-1| \Big|_{b}^{-2} - \lim_{b \to -\infty} \ln|x+1| \Big|_{b}^{-2}$$

$$= \lim_{b \to -\infty} \ln\left|\frac{x-1}{x+1}\right| \Bigg|_{b}^{-2}$$

$$= \lim_{b \to -\infty} \ln\left(\left|\frac{-3}{-1}\right| - \ln\left|\frac{b-1}{b+1}\right|\right)$$

$$= \ln 3 - \ln\left(\lim_{b \to -\infty} \frac{b-1}{b+1}\right) = \ln 3 - \ln 1 = \ln 3$$

b) $$\int_{1}^{2} \frac{1}{x(\ln x)^p}\, dx \quad ; \quad t = \ln x$$

$$= \int_{0}^{\ln 2} \frac{dt}{t^p} = \lim_{b \to 0^+} \left[\frac{1}{-p+1} t^{1-p}\right]_{b}^{\ln p}$$

$$= \lim_{b \to 0^+} \left(\frac{b^{1-p}}{p-1} + \frac{1}{1-p} (\ln 2)^{1-p}\right)$$

the integral converges for $p<1$ and diverges for $p \geq 1$.